

Nonequilibrium Kondo model: Crossover from weak to strong coupling

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We analyze the nonequilibrium Kondo model at finite voltage and temperature by using a new formulation of the real-time renormalization group method with the Laplace variable as the flow parameter. We evaluate the energy-dependent spin relaxation rate and nonlinear conductance, and derive an approximate form for the universal line shape for the latter in the whole crossover regime from weak to strong coupling. The results are shown to agree well with exact methods in equilibrium, Fermi-liquid theory, weak-coupling expansions, and recent experiments. For the transient spin dynamics we find a universal exponential decay in the long-time limit along with a truncation-dependent pre-exponential power law. For multichannel models a pure power-law decay typical for non-Fermi-liquid behaviour is predicted.

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The solution of the nonequilibrium Kondo model is a basic unsolved problem in condensed matter physics. In its elementary version, the model consists of a spin $\frac{1}{2}$ coupled antiferromagnetically to two fermionic metallic reservoirs. It reveals the Kondo effect of complete spin screening at low energies and plays an important role in bulk systems with magnetic impurities [1]. In the Coulomb blockade regime of single-level quantum dots, the Kondo model can be realized via its relation to the symmetric Anderson impurity model [2]. In this case the effective exchange coupling is given by $J \sim \frac{2\Delta}{\pi U}$, where Δ is the level broadening and $U \sim D$ is the charging energy, defining the band width $2D$. This allows for a controlled experimental realization of the Kondo model in nonequilibrium, where the Kondo effect has been observed via a zero-bias anomaly [3]. In equilibrium, the Kondo model has been solved by a Bethe ansatz [4] and conformal field theory (CFT) [5]. Using the numerical renormalization group (NRG) method [6–8] and a Bethe ansatz [9], the temperature dependent linear conductance $G(T)$ has been shown to be a universal function of T/T_K , where $T_K \sim De^{-1/(2J)}$ is the Kondo temperature. Analytical results are available for low ($T \ll T_K$) and high ($T \gg T_K$) temperatures from Fermi-liquid theory and poor man scaling methods [10–12]. A challenging issue is the nonequilibrium properties at $T = 0$, such as the determination of the nonlinear conductance $G(V)$ as a function of bias voltage V and the transient spin dynamics $\langle \underline{S} \rangle(t)$. The weak-coupling regime $\max\{V, 1/t\} \gg T_K$ has been solved by improved poor man scaling methods using phenomenological decay rates [13–15], flow equations [16, 17], and the real-time renormalization group (RTRG) method [18–20]. However, the analysis of universal properties in the strong coupling regime $V, 1/t \lesssim T_K$ remains an open problem.

In this Letter we propose an approximate solution of this problem by using a new formulation of the RTRG method where the Laplace variable E is used as the flow

parameter (called “ E -flow” in the following). Within this scheme the relaxation rates occurring in each renormalization group (RG) step are the full physical ones, and their energy dependence appears to be crucial for the description of the crossover from weak to strong coupling. Applying the E -flow to the nonequilibrium Kondo model, we calculate the nonlinear conductance and observe that both the T and V dependence agree well with recent experiments [21]. We show analytically that in the scaling limit ($D \rightarrow \infty$, $J \rightarrow 0$, $T_K = \text{const}$) *both* the low and high energy properties are consistent with Fermi-liquid theory [11] and weak-coupling expansions, respectively. This distinguishes our approach from other studies of the nonequilibrium Anderson impurity model (see, *e.g.*, Refs. 22 and 23 for reviews), where the effective spin exchange coupling J cannot be chosen arbitrarily small to achieve universality in the scaling limit. Also approximate calculations of $G(V)$ via the Bethe ansatz eigenstates [9] cannot cover the crossover to the weak coupling regime. We find that our $G(T)$ curve agrees within a few percent with the NRG one and that the nonlinear conductance $G(V)$ is almost independent of the truncation order.

Finally we address the transient spin dynamics after an initial coupling of the spin to the bath. In Ref. [20] it has been predicted in the weak coupling regime that the long-time dynamics is always exponential, accompanied by pre-exponential power laws in higher orders. Scaling behavior has been observed within time-dependent NRG calculations [24] and exponential behavior was predicted at the Toulouse point [25] and in the framework of integrable quantum field theories [26]. However, due to the presence of branch-cut contributions the transient spin dynamics is still under debate (see Ref. [27] for a discussion within the spin boson model). At $T = V = 0$ we find a universal exponential decay in the long-time limit $t \gg 1/T_K$ with a rate $\Gamma^* \sim T_K$, and, in addition, a pre-exponential power-law behavior with

a truncation-dependent exponent. In contrast, for multi-channel Kondo models, we predict a pure power-law decay in the long time limit as a result of non-Fermi-liquid behaviour at low energies [5, 28].

Model.—We consider a single spin $\frac{1}{2}$ with spin \underline{S} , which is coupled by an isotropic exchange $H_{ex} = J_0 \underline{S} \cdot \underline{s}$ to the spins $\underline{s} = \frac{1}{2} \sum_{\alpha\alpha'\sigma\sigma'kk'} a_{\alpha\sigma k}^\dagger \underline{\sigma}_{\sigma\sigma'} a_{\alpha'\sigma'k'}$ of two reservoirs labeled by $\alpha = L, R$. $\sigma = \downarrow, \uparrow$ denotes the spin, k is the state index and $\underline{\sigma}$ are the Pauli matrices. The reservoirs are described by $H_{res} = \sum_{\alpha\sigma k} \epsilon_{\alpha k} a_{\alpha\sigma k}^\dagger a_{\alpha\sigma k}$ with finite band width and are kept on chemical potentials $\mu_{L/R} = \pm V/2$ (we use units $e = \hbar = k_B = 1$).

Liouvillian approach.—Following Ref. 18, we describe the dynamics of the reduced density matrix $\rho(t)$ of the local spin by the effective von Neumann equation $\dot{\rho}(t) = -i \int_0^t dt' L(t-t')\rho(t')$, which reads in Laplace space $\rho(E) = i/(E - L(E))\rho_0$ [29]. At the initial time $t = 0$ the system is assumed to factorize into an arbitrary local part ρ_0 and a grand canonical equilibrium part $\rho_L^{\text{eq}} \rho_R^{\text{eq}}$ for the reservoirs. $L(E)$ is an effective Liouvillian, which results from integrating out the reservoir degrees of freedom. Using spin conservation and rotational invariance for the isotropic Kondo model we find that the Liouvillian can be parameterized by the spin relaxation rate $\Gamma(E)$ via the nonvanishing matrix elements $L(E)_{ss',s'} = -\frac{i}{2}ss'\Gamma(E)$ and $L(E)_{s,-s,s,-s} = -i\Gamma(E)$, where $s = \uparrow, \downarrow \equiv \pm$. The Liouvillian has a zero eigenvalue corresponding to the stationary state and three degenerate eigenvalues $-i\Gamma(E)$ describing the spin dynamics via

$$\langle \underline{S} \rangle(E) = \frac{i}{E + i\Gamma(E)} \langle \underline{S} \rangle(t=0), \quad (1)$$

from which the real-time dynamics can be obtained via the inverse Laplace transform. Analogously, one can also derive an equation for the current $I_\alpha(E) = -i\text{Tr}\Sigma_\alpha(E)\rho(E)$ flowing into the reservoir α , where $\Sigma_\alpha(E)_{ss',s'} = i\Gamma_\alpha(E)$ are the nonvanishing matrix elements of the current kernel. Defining $G(E) = \pi \frac{\partial}{\partial V} \Gamma_L(E)$, the stationary conductance in units of $G_0 = 2e^2/h$ is given by $G = G(0)$.

RG formalism.—To calculate $\Gamma(E)$ and $G(E)$ we use a different formulation of the RTRG method than the one proposed in Ref. 18. Instead of introducing an artificial cutoff Λ on the imaginary frequency axis, we use the physical Laplace variable E itself as the flow parameter. This means that we differentiate the full diagrammatic series for the effective kernels and vertices w.r.t. E and obtain a self-consistent set of equations in terms of effective quantities by an appropriate resummation of all diagrams (see details in the Supplementary Material [30]). It turns out that the scaling limit is well-defined if two E -derivatives are taken for the kernels, whereas a single derivative is sufficient for the vertices. The exact RG equations are finally truncated systematically by considering all diagrams up to a certain order in

the effective vertices but keeping the full propagator between the vertices. The reliability of our approach in the strong coupling regime is tested by comparing the results in second and third order truncation. These truncation schemes are essentially different from conventional 1- and 2-loop treatments, since we consider the full propagator between the vertices and since we use the same truncation scheme for the rate $\Gamma(E)$ including its full energy dependence. We illustrate this by showing the RG equations for $T = V = 0$ (the full equations at finite T, V can be found in the Supplementary Material [30]). Neglecting all terms of $O(J^4)$, we obtain with $E = i\Lambda$

$$d_\Lambda^2 \Gamma = -\frac{4J^2}{\Lambda + \Gamma} \quad , \quad d_\Lambda J = -\frac{2J^2(1 + ZJ)}{\Lambda + \Gamma} \quad , \quad (2)$$

where $Z = \frac{1}{1 + \frac{4J}{\Lambda}}$ denotes the Z -factor. With $\lambda = \Lambda + \Gamma$ the RG equation for the conductance reads $d_\Lambda G = -\frac{3\pi^2}{2\lambda} J_I K$, which is coupled to two other vertices with the RG equations $d_\Lambda J_I = -2J_I J/\lambda$ and $d_\Lambda K = -4KJ(1 - ZJ)/\lambda$. With $\tilde{J} = ZJ$ we can write Eq. (2) in the form $d_\lambda \tilde{J} = -2\tilde{J}(1 - \tilde{J})/\lambda$ which gives the invariant $T_K = (\Lambda + \Gamma) \sqrt{\frac{\tilde{J}}{1 - \tilde{J}}} e^{-\frac{1}{2\tilde{J}}}$, defining the Kondo temperature. In contrast, the standard 2-loop Kondo temperature is obtained by neglecting the relaxation rate Γ leading to the poor man scaling solution $J_p(\Lambda)$, defined by $T_K = \Lambda \sqrt{\frac{J_p}{1 - J_p}} e^{-\frac{1}{2J_p}}$. Thus our approach provides a microscopic foundation for the physically intuitive picture of the relaxation rate cutting off the RG flow, including its full energy dependence. As an important consequence we find in our approach that the 2-loop fixed point at $\Lambda = 0$ is shifted to the negative value $\Lambda = -\Gamma^* < 0$. As shown below this is fundamentally related to the exponential relaxation of the spin dynamics. Moreover, a technical advantage is that the point $E = i\Lambda = 0$, defining the stationary value of all physical quantities, is well separated from the fixed point. This allows for a well-defined expansion around this point leading to the correct Fermi-liquid relations (see below).

The initial conditions at high energies $\Lambda_0 \gg T_K$ are given by $J = J_0$, $Z = 1$, $K = 2J_0^2$, $J_I = J_0$ and $G = \frac{3\pi^2}{4} J_0^2$ [18]. In the scaling limit, this leads to the solution $Z = (1 - \tilde{J})^2$ and $G = \frac{3\pi^2}{4} \tilde{J}^2$. For Γ a special procedure is needed since it is proportional to Λ in the weak coupling regime $\Gamma = 2\Lambda(J_p + O(J_p^2))$. This leads to a truncation order independent instability for its value in the regime $\Lambda \ll T_K$. We solve this problem by complementing our approach with the exactly known result of unitary conductance $G(E=0) = \frac{3\pi^2}{4} \tilde{J}^2(E=0) = 1$, which is a consequence of the Friedel sum rule [1]. This fixes the values $\tilde{J} \equiv \tilde{J}(0) = \frac{2}{\pi\sqrt{3}} \approx 0.37$, $\bar{\Gamma} \equiv \Gamma(0) = \sqrt{\frac{1 - \tilde{J}}{\tilde{J}}} e^{\frac{1}{2\tilde{J}}} T_K \approx 5.11 T_K$, and $\bar{Z} = (1 - \tilde{J})^2 \approx 0.40$, and the RG equations (2) can be solved numerically starting from $\Lambda = 0$. The $T = V = 0$ results for all quanti-

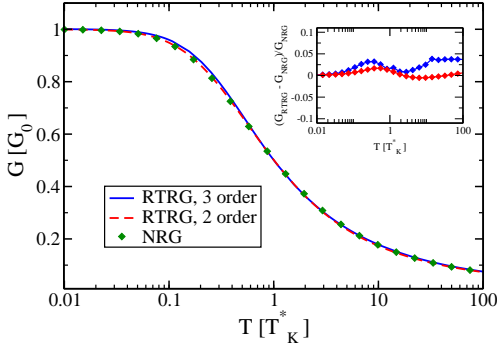


FIG. 1. (Color online) Main panel: $G(T)$ in second (red dashed line) and third (blue full line) order truncation as function of T/T_K^* , defined by $G(T = T_K^*) = 1/2$, compared to the NRG results (green diamonds). Inset: $(G_{\text{RTRG}} - G_{\text{NRG}})/G_{\text{NRG}}$ as function of T/T_K^* in second (lower line) and third (upper line) order truncation.

ties at large $\Lambda_0 \gg T_K$ are then used as initial condition to solve the RG equations at finite T, V using an initial value $\Lambda_0 \gg T, V, T_K$.

Finite T.—As a benchmark for the reliability of our approach we first compare the linear conductance $G(T)$ with the NRG result. When plotted in units of T_K^* , defined by $G(T = T_K^*) = 1/2$, we see in Fig. 1 that our results in second and third order truncation fall almost on top of each other and agree very well with the NRG result [6]. The low- and high-temperature regimes can be studied analytically. For $T \gg T_K^*$, we expand in $J_p(T) \ll 1$ and get

$$G(T) \approx \frac{3\pi^2}{4} J_p^2(T) \left(1 - J_p(T) 4 \ln \frac{2\pi}{e^{1+\gamma}} + O(J_p^2) \right), \quad (3)$$

where $\gamma = 0.577 \dots$ is Euler's constant. Besides the standard weak-coupling result, the second term is a third order correction which suppresses the conductance. For $T \ll T_K$, we expand in \bar{J} and obtain in third order truncation the Fermi-liquid result $G(T) \approx 1 - c_T (\frac{T}{\bar{\Gamma}})^2$ with $c_T = \pi^4 \frac{\bar{J}^3(1+3\bar{J})}{(1-\bar{J})^3} + O(\bar{J}^5)$. For the absolute value of the curvature w.r.t. T_K^* we obtain in third order truncation $c_T (T_K^*/\bar{\Gamma})^2 \approx 4.8$, which is much better than the second order result but still not fully reliable due to the inaccuracy in the ratio $T_K^*/\bar{\Gamma}$ (the NRG result is about 10% larger [31]).

Finite V.—Next we consider $T = 0$ but finite V . Figure 2 shows again that our second and third order truncation results for $G(V)$ agree very well with each other when plotted in units of T_K^{**} defined by $G(V = T_K^{**}) = 1/2$. For large $V \gg T_K^{**}$ we expand in $J_p(V) \ll 1$ and obtain

$$G(V) \approx \frac{3\pi^2}{4} J_p^2(V) + O(J_p^4) \quad . \quad (4)$$

The first term is the standard weak-coupling result, but in addition our third order analysis proves the absence

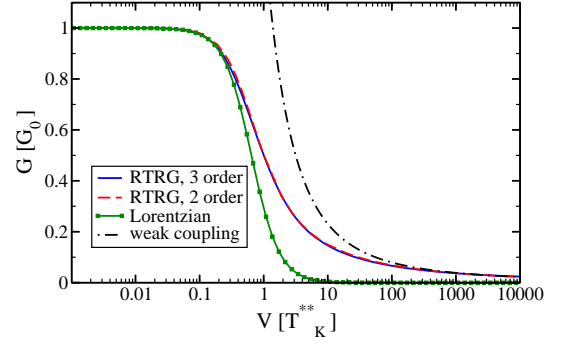


FIG. 2. (Color online) $G(V)$ in second (red dashed line) and third (blue full line) order truncation as function of T/T_K^{**} , defined by $G(V = T_K^{**}) = 1/2$, compared to Lorentzian fit (green squares) and high- V result (4) (dotted-dashed line).

of a term in $O(J_p^3)$. For small $V \ll T_K^{**}$, we get $G(V) \approx 1 - c_V (\frac{V}{\bar{\Gamma}})^2$ with $c_V/c_T = 3/(2\pi^2)$. This coincides with the exact ratio known from Fermi-liquid theory [11, 32]. The result is fulfilled order by order by expanding in \bar{J} . The numerical solution does not show the exact ratio since inconsistent terms of $O(\bar{J}^5)$ are involved there, but in third order truncation the error is already below 10%.

In Fig. 2 we show a comparison with a Lorentzian fit function $G(V) = 1/[1 + c_V(V/\bar{\Gamma})^2]$, as proposed in Ref. [9] from Bethe ansatz calculations. It turns out to be a good fit only for voltages significantly below T_K^{**} , but becomes inconsistent with the weak-coupling solution when applied in the whole crossover regime. The alternative trial function $G(V) = [1 + (V/T_K'(x))^2]^{-s}$ with $T_K'(x) = T_K^{**} (\frac{1-b+bx^{s'}}{2^{1/s}-1})^{1/2}$ and $x = V/T_K^{**}$, which is a good fit for $G(T)$ at $s = 0.22$, $b = 0$ and $T_K^* \rightarrow T_K^{**}$ [6], turns out to describe the crossover regime of $G(V)$ quite well at the significantly different values $s = 0.32$, $b = 0.05$ and $s' = 1.26$. This shows that the universal line shapes of $G(V)$ and $G(T)$ are quite different and cannot be rescaled on top of each other, as shown in Fig. 3 when plotted in terms of the same unit T_K^* . A numerical analysis gives $T_K^{**}/T_K^* \sim 1.8$ and we find that, in third order truncation, $G(V)$ lies always above $G(T)$ for $V = T$, which can already be anticipated from our analytical results at low and high energies. In Fig. 3 we also show that recent experiments appear to be well captured by the present theory. The experiments are performed on an InAs nanowire quantum dot [21], which can be represented as a single-level dot with the effective spin exchange $J \sim \frac{2\Delta}{\pi U} \sim 0.28$. The experimental data for $G(T)$ and $G(V)$ fit our results quite well in the regime $T, V \lesssim T_K^*$ using the *same* rescaling factor for the V and T dependence. Since $T_K^* \sim 0.1 U$ in this experiment, universal features from spin fluctuations can only be expected below T_K^* , while above T_K^* charge fluctuations will set in. As proposed in Ref. [33],

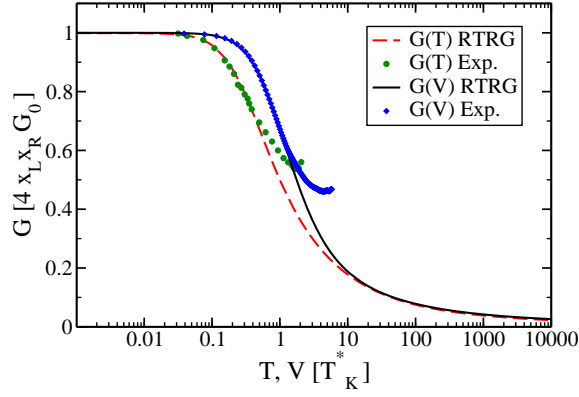


FIG. 3. (Color online) Comparison between the RTRG results and experimental data [21] for $G(T)$ and $G(V)$. Both T and V are expressed in units T_K^* defined by $G(T = T_K^*) = 1/2$. An asymmetry factor $4x_L x_R \approx 0.8$ is introduced to match the experimental $G(0)$ with $4x_L x_R$.

the universal relation $G(V = T_K^*) \approx \frac{2}{3}$ allows for an elegant experimental determination of the important energy scale T_K^* . The experimental conductance is not unitary due to uncontrolled asymmetric couplings to the two leads. We have checked that asymmetric couplings $J_{\alpha\alpha'} = 2\sqrt{x_\alpha x_{\alpha'}}J$ with $0 < x_\alpha < 1$ and $x_L + x_R = 1$ only give rise to the well-known asymmetry factor $4x_L x_R$ for the conductance in the weak and strong coupling regime [19, 32] and change the normalized conductance $G(V)/(4x_L x_R)$ in the intermediate voltage regime by less than 1% for the experimental value $4x_L x_R \sim 0.8$. Furthermore, we checked that the finite experimental temperature $T \sim 10 \text{ mK} \sim 0.025 T_K^*$ does not change $G(V)$ significantly.

Time evolution.—At $T = V = 0$ and for $t \gg 1/T_K$ we find from Eqs. (1) and (2) that the spin dynamics is given by $\langle \underline{S} \rangle(t) \sim \frac{1}{t^g} e^{-\Gamma^* t}$. The resolvent $1/[E + i\Gamma(E)]$ has a branch cut on the negative imaginary axis with a branching pole at $E = -i\Gamma^*$, which is identical to the fixed point $\tilde{J} = J^*$ of the RG flow and determines the rate of the exponential decay. We obtain the nearly constant ratio $\Gamma^*/T_K \approx 1.1$ independent of the truncation order. To determine the exponent g we expand around the fixed point and set $E = i\Lambda$ with $|\Lambda + \Gamma^*| \ll \Gamma^*$. We obtain $\frac{dZ}{d\Lambda} = 4Z\tilde{J}^2/\lambda \approx \alpha Z/\lambda$ with $\alpha = 4J^{*2}$. This gives $Z \sim \lambda^\alpha$ and, by using $d_\Lambda \lambda = Z^{-1} \sim \lambda^{-\alpha}$, we get $\lambda \sim (\Lambda + \Gamma^*)^{1/(1+\alpha)}$, leading to $g = \frac{\alpha}{1+\alpha} = \frac{4J^{*2}}{1+4J^{*2}}$ after the inverse Laplace transform. However, the fixed point value J^* depends on the truncation order. In second (third) order truncation we get $J^* = 1$ and $g = \frac{4}{5}$ ($J^* = \frac{1}{2}$ and $g = \frac{1}{2}$). As a result we obtain a universal exponential decay but the pre-exponential power law cannot be determined unambiguously.

In contrast, for N -channel Kondo models with $N > 1$, where $\frac{dZ}{d\Lambda} = 4NZ\tilde{J}^2/\lambda$ and $\frac{d\tilde{J}}{d\Lambda} = -2\tilde{J}^2(1 - N\tilde{J})/\lambda$, the non-Fermi-liquid behavior at low energies [5, 28] requires

the fixed point to be at $E = 0$. This gives rise to a pure power-law decay for the spin dynamics in the long-time limit. The exponent follows from $\alpha = 4NJ^{*2}$ and $J^* = \frac{1}{N}$ to be $g = \frac{4NJ^{*2}}{1+4NJ^{*2}} = \frac{4}{4+N}$. However, the precise value of the exponent is only reliable for $N \gg 1$, where the β -function can be systematically truncated [34].

Conclusions.—We have derived an approximate form for the universal line shape of the zero-bias anomaly for the isotropic Kondo model, which successfully describes the recent experiments [21, 33]. The reliability of our approach has been checked by comparing different truncation orders and by reproducing the correct asymptotic values at large and small voltages. In the long-time limit we found a universal exponential decay for the spin dynamics in the 1-channel case and a power-law decay for multichannel models. Our approximate results could serve as a reference for future more refined numerical and analytical approaches. In particular, the precise value of the exponential decay rate and the form of the pre-exponential modulation is an open issue. The new formulation of the RTRG method using the Laplace variable as the flow parameter is a promising analytical tool for solving other nonequilibrium problems of quantum impurity physics and dissipative quantum mechanics in the strong coupling regime.

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Nonequilibrium Kondo model: Crossover from weak to strong coupling Supplementary Material

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The following appendices comprise the supplementary material to Ref. 1.

I. THE E -FLOW SCHEME OF RTRG

In Ref. 2 a real-time renormalization group method (RTRG) has been developed in Liouville space, which can be used to evaluate the effective Liouvillian $L(E)$ of a local quantum system coupled to several fermionic reservoirs. Within this method, a Matsubara cutoff was defined by cutting off the Matsubara poles of the Fermi functions. A systematic weak-coupling expansion including cutoff scales from relaxation and decoherence rates was developed, which was used to calculate the nonequilibrium properties of the Kondo model³ and the interacting resonant level model (IRLM)⁴ in the weak coupling regime. In the strong coupling regime and in the presence of spin, orbital or potential fluctuations, a problem within the RTRG method is the generation of spurious terms linear in the physical band width $2D$ of the reservoirs, originating from the fact that the cutoff used in the RG is not identical to the physical band width. In particular, it turns out that the RG-flow is unstable against a small variation of these terms. In the weak-coupling regime it has been shown^{2,3} that the problem can be resolved by means of an appropriate subtraction scheme. The same was demonstrated for the strong coupling regime of the IRLM⁴. However, in the generic strong coupling case it is not clear whether the subtraction scheme can be uniquely defined.

Here we propose an alternative flow scheme where this problem is avoided by solving the RG-equations starting at low energies. It uses the Laplace variable E as flow parameter and for this reason is called E -flow scheme in the following. Another advantage of the E -flow scheme consists in the fact that the Laplace variable is a physical energy scale from which the time evolution can be directly evaluated. The effective Liouvillian and the vertices occurring in the RG equations are the full physical ones. Furthermore, the method allows the consideration of all diagrams up to a certain order without adhering to approximations typically used in the weak-coupling regime.

We show here the derivation of the E -flow RG equations in third order truncation for the case where only fermionic 2-point vertices $G_{12}(E, \bar{\omega}_1, \bar{\omega}_2)$ are present and describe either spin, orbital or potential fluctuations (higher-order vertices are only generated in fourth order). Moreover, we assume that the bare vertices have no explicit frequency-dependence and we take a flat d.o.s. in the reservoirs with band width $2D$. The index $1 \equiv \eta\alpha\sigma$ characterizes the reservoir field operator: $\eta = \pm$ distinguishes creation/annihilation operators, α is the reservoir index, and σ is the channel index (i.e. the spin for the Kondo model). The frequency arguments $\bar{\omega}_i = \eta_i\omega_i$ contain the reservoir energy ω_i measured relative to the chemical potential μ_α of reservoir α . The Keldysh index p can also be included in the index 1 but is disregarded here since, as will be shown below, only the average over the Keldysh indices remains in the final RG equations. Including the Keldysh indices the contraction between two vertices is given by $\gamma_{11'}^{pp'}(\omega, \omega') = \delta_{1\bar{1}'}\delta(\bar{\omega} + \bar{\omega}')\gamma^{p'}(\bar{\omega})$, where $\gamma^{p'}(\bar{\omega}) = p'f(p'\bar{\omega})\frac{D^2}{D^2 + \bar{\omega}^2}$, $1 = \eta\alpha\sigma$, $\bar{1}' = -\eta'\alpha'\sigma'$, and $f(\omega) = \frac{1}{e^{\omega/T} + 1}$ denotes the Fermi function at temperature T .

To achieve convergence of the frequency integrals in the scaling limit $D \rightarrow \infty$ one needs equations for the second derivative of $L(E)$ and the first derivative of $G_{12}(E, \bar{\omega}_1, \bar{\omega}_2)$ w.r.t. E . Using the diagrammatic representation described in Ref. 2, the E -derivative can only act on the propagators $\Pi(E) = \frac{1}{E - L(E)}$ occurring between the bare vertices (we always resum all self-energy insertions such that the full effective Liouvillian appears in the propagator). Resumming all diagrams right and left to the propagator in which the E -derivative is taken, we obtain the following two equations up to $O(G^3)$

$$\frac{1}{2} \frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \frac{1}{2} \text{ (diagram 1) } + \text{ (diagram 2) } + O(G^4) \quad , \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial E} G_{12}(E, \bar{\omega}_1, \bar{\omega}_2) = & \left(\text{diagram 1} - (1 \leftrightarrow 2) \right) + \frac{1}{2} \text{ (diagram 2) } + \frac{1}{2} \text{ (diagram 3) } \\ & + \left(\text{diagram 4} + \text{diagram 5} - (1 \leftrightarrow 2) \right) + O(G^4) \quad . \quad (2) \end{aligned}$$

Here the slash indicates the E -derivative $\frac{\partial}{\partial E}$ and a double-circle represents the full effective 2-point vertex (containing all connected diagrams with two free reservoir lines). Symmetry factors $\frac{1}{n!}$ arising either from the diagrammatic rules (when two vertices are connected by n equivalent lines) or from the E -derivatives $\frac{1}{n!} \frac{\partial^n}{\partial E^n}$ are explicitly quoted. Since the propagators $\Pi_{1\dots n} = \Pi(E_{1\dots n} + \bar{\omega}_{1\dots n})$ between the vertices contain the variables $E_{1\dots n} = E + \bar{\mu}_{1\dots n} = E + \bar{\mu}_1 + \dots + \bar{\mu}_n$ and $\bar{\omega}_{1\dots n} = \bar{\omega}_1 + \dots + \bar{\omega}_n$ with $\bar{\mu}_i = \eta_i\mu_{\alpha_i}$ and $\bar{\omega}_i = \eta_i\omega_i$, the E -derivative can be written as frequency-derivative $\frac{\partial}{\partial \bar{\omega}_i}$ and we can apply partial integration to calculate the frequency integrals. E.g., for the first term on the r.h.s. of (2) we

obtain (we have permuted the two indices of the vertices by using antisymmetry $G_{12}(E, \bar{\omega}_1, \bar{\omega}_2) = -G_{21}(E, \bar{\omega}_2, \bar{\omega}_1)$)

$$\begin{aligned} \text{Diagram 1} &= - \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} \\ &= - \text{Diagram 5} - \text{Diagram 6} - \text{Diagram 7} + O(G^4) \end{aligned} \quad (3)$$

Here the cross indicates the frequency derivative w.r.t. $\bar{\omega}$ of either the corresponding reservoir contraction or the frequency argument of the vertices. The first line is exact and follows from partial integration. For the derivation of the second line we have used

$$\text{Diagram 8} = \text{Diagram 9} + O(G^3), \quad \text{Diagram 10} = \text{Diagram 11} + O(G^3), \quad (4)$$

which follows analogously to (2) by using the fact that, in the original diagrammatic series, a frequency associated with an external line occurs only in those propagators which lie below the external line but not in other propagators and not in the bare vertices. Note that these relations can only be applied if it is specified whether the external line involving the frequency derivative is directed either towards the left or towards the right.

Analogously, we can treat the first term on the r.h.s. of (1) by two partial integrations and using the fact that $\frac{\partial^2}{\partial \bar{\omega}_1 \partial \bar{\omega}_2} G_{12}(E, \bar{\omega}_1, \bar{\omega}_2) \sim O(G^3)$. We obtain

$$\begin{aligned} \text{Diagram 12} &= \text{Diagram 13} + 2 \text{Diagram 14} + 2 \text{Diagram 15} + O(G^4) \\ &= \text{Diagram 16} - 4 \text{Diagram 17} + O(G^4), \end{aligned} \quad (5)$$

where, in the second step, we have again used (4).

Inserting (5) and (3) in (1) and (2), respectively, we see that many diagrams in $O(G^3)$ cancel each other and we cast the final RG-equations to a very compact and generic form

$$\frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \text{Diagram 18} + O(G^4), \quad (6)$$

$$\frac{\partial}{\partial E} G_{12}(E, \bar{\omega}_1, \bar{\omega}_2) = - \left(\text{Diagram 19} - (1 \leftrightarrow 2) \right) - \frac{1}{2} \text{Diagram 20} + O(G^4). \quad (7)$$

The RG equations are constructed in such a way that all frequency integrals on the r.h.s. of the RG equations are well-defined in the limit $D \rightarrow \infty$ even when the frequency dependence of the vertices is neglected. Performing this limit, each reservoir contraction contains $\gamma^{p'}(\bar{\omega}) = p' f(p' \bar{\omega})$. As a consequence, the frequency integral for a contraction where the derivative is taken (diagrammatically indicated by the cross) involves $\frac{\partial}{\partial \bar{\omega}} \gamma^{p'}(\bar{\omega}) = f'(\bar{\omega})$, which is independent of the Keldysh indices. The frequency integral of the contraction without a derivative in the second term on the r.h.s. of (7) involves $\gamma^{p'}(\bar{\omega}) = f^a(\bar{\omega}) + \frac{p'}{2}$, where $f^a(\omega) = f(\omega) - \frac{1}{2}$ is the antisymmetric part of the Fermi function. Since two propagators are involved in this frequency integral, the integration contour can be closed in the upper half of the complex plane and the symmetric part $\frac{p'}{2}$ vanishes identically since the resolvents and the vertices are analytic functions in the upper half plane². Therefore, an explicit dependency on the Keldysh indices disappears, and it is sufficient to consider the vertices $G_{12}(E, \bar{\omega}_1, \bar{\omega}_2) = \sum_{p_1 p_2} G_{12}^{p_1 p_2}(E, \bar{\omega}_1, \bar{\omega}_2)$ averaged over the Keldysh indices. We note that the symmetric part of the Fermi function does not influence the RG equations but enters into the initial conditions at high energies².

To evaluate the frequency integrals on the r.h.s. of the RG equations (6) and (7) explicitly, we need a consistent approximation for the frequency dependence of the vertices and the Liouvillian. For the vertices, this can be achieved by formally integrating the first two terms on the r.h.s. of (7) yielding

$$\text{Diagram 21} = \text{Diagram 22} + \text{Diagram 23} - \text{Diagram 24} + O(G^3), \quad (8)$$

where $G_{12}(E) = G_{12}(E, 0, 0)$ is indicated by filled double dots. The open circle at a contraction indicates that the propagator corresponding to the vertical cut at the position of that circle has to be replaced by the difference $\Pi(E_{1\dots n} + \bar{\omega}_{1\dots n} + \bar{\omega}) - \Pi(E_{1\dots n} + \bar{\omega}_{1\dots n})$ of the two propagators with and without the frequency $\bar{\omega}$ corresponding to the contraction with the circle. The identity (8) refers to the case where the two external lines are directed to the right, similar equations can be written down for the other cases. We note that the sign factor in the second term on the r.h.s. accounts explicitly for the crossing of the two external lines relative to the first term. Therefore, when using

(8) in a certain diagram the sign factor must not be written explicitly since it is automatically accounted for in the diagrammatic rules. Inserting (8) in (6) and (7) we obtain

$$\frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + O(G^4), \quad (9)$$

$$\begin{aligned} \frac{\partial}{\partial E} G_{12}(E) = & - \left(\text{diagram 4} + \text{diagram 5} + \text{diagram 6} - (1 \leftrightarrow 2) \right) \\ & - \frac{1}{2} \text{diagram 7} + O(G^4) . \end{aligned} \quad (10)$$

Finally, to calculate the frequency integrals in these equations, one needs a consistent approximation for the propagator $\Pi(E + \bar{\omega})$, where $\bar{\omega}$ is an integration variable. Using $\frac{\partial^2}{\partial E^2} L(E) \sim O(\frac{1}{E} G^2)$ cf. (9), we approximate $L(E + \bar{\omega}) \approx L(E) + \frac{\partial}{\partial E} L(E) \bar{\omega}$. This leads to the following approximation for the propagator

$$\Pi(E + \bar{\omega}) \approx \frac{1}{\bar{\omega} + \chi(E)} Z(E) , \quad (11)$$

where $Z(E) = \frac{1}{1 - \frac{\partial}{\partial E} L(E)}$ is the Z -factor operator and $\chi(E) = Z(E)(E - L(E))$ is an operator defining the distance to resonance positions. We note that the eigenvalue zero of the Liouvillian $L(E)$ (or of $Z(E)L(E)$) can be omitted in (11) since the projector P_0 on the corresponding eigenvector fulfils $P_0 L(E) = P_0 G_{12} = 0$ and $P_0 Z(E) = 1^2$. Using (11) in (9) and (10), all frequency integrals can be calculated analytically, which will be done explicitly for the Kondo model in the next section.

Equations (9) and (10) along with (11) are the final RG equations for a determination of the effective Liouvillian $L(E)$ from which the reduced density matrix of the local system in Laplace space follows via $\rho(E) = \frac{i}{E - L(E)} \rho_0$, where $\rho_0 = \rho(t=0)$ is the initial condition. To find the average of the current $I_\alpha = -i \text{Tr} \Sigma_\alpha(E) \rho(E)$ flowing into reservoir α , one needs the current kernel $\Sigma_\alpha(E)$ in Laplace space. As shown in Ref. 2 it can be determined analogously to $L(E)$ by replacing the first vertex from the left in all diagrams of (9) and (10) by the current vertex $I_{12}^\alpha(E)$. To calculate the differential conductance one needs the variation δI_α for an infinitesimal variation $\delta \mu_\alpha$ of the chemical potentials of the reservoirs. Within the E -flow scheme the RG equation for $\frac{\partial}{\partial E} \delta L(E)$ (or, equivalently, for $\frac{\partial}{\partial E} \delta \Sigma_\alpha(E)$ by replacing the first vertex by the current vertex) can be straightforwardly established by applying the variation to the original diagrammatic series and using $\delta \Pi_{1\dots n} = \delta \bar{\mu}_{1\dots n} \frac{\partial}{\partial E} \Pi_{1\dots n} + \Pi_{1\dots n} \delta L_{1\dots n} \Pi_{1\dots n}$, where $\delta L_{1\dots n} = (\delta L)(E_{1\dots n} + \bar{\omega}_{1\dots n})$. After resummation we obtain by analogy with (1) up to $O(G^3)$

$$\begin{aligned} \frac{\partial}{\partial E} \delta L(E) = & \frac{1}{2} \delta \bar{\mu}_{12} \text{diagram 8} + \frac{1}{2} \text{diagram 9} + \frac{1}{2} \text{diagram 10} \\ & + (\delta \bar{\mu}_{12} + \delta \bar{\mu}_{13}) \text{diagram 11} + O(G^4) , \end{aligned} \quad (12)$$

where $\delta L \sim O(\delta \mu G)$ is represented by diagram 12 and can be calculated from the first (lowest order) term in the r.h.s. of (12) and subsequently inserted in the second and third term in the r.h.s. of (12). Applying two partial integrations to the first term on the r.h.s. of (12) by analogy with (5), we obtain

$$\frac{\partial}{\partial E} \delta L(E) = \frac{1}{2} \delta \bar{\mu}_{12} \text{diagram 13} - \frac{1}{2} \text{diagram 14} + O(G^4) . \quad (13)$$

Using (8) we get the final RG equation for the variation of the kernel

$$\begin{aligned} \frac{\partial}{\partial E} \delta L(E) = & \frac{1}{2} \delta \bar{\mu}_{12} \text{diagram 15} - \frac{1}{2} \text{diagram 16} \\ & + \delta \bar{\mu}_{13} \text{diagram 17} + \delta \bar{\mu}_{12} \text{diagram 18} + O(G^4) . \end{aligned} \quad (14)$$

We note that at zero temperature all diagrams in (9), (10) and (14) containing a contraction with a circle and a cross vanish, since the cross restricts the frequency to zero value.

II. RG EQUATIONS FOR THE KONDO MODEL

We now apply the RG-equations (9), (10) and (14) to the isotropic spin- $\frac{1}{2}$ Kondo model at zero magnetic field. Using spin conservation and rotational invariance it has been shown in Ref. 2 how the kernels and the vertices can be decomposed into convenient basis superoperators. One obtains $L(E) = -i\Gamma(E)L^a$, $\Sigma_\gamma(E) = i\Gamma_\gamma(E)L^b$, $G_{+\alpha\sigma,-\alpha'\sigma'}(E) = \sum_{\chi=a,2,3} G_{\alpha\alpha'}^\chi(E)\hat{L}_{\sigma\sigma'}^\chi$, and $I_{+\alpha\sigma,-\alpha'\sigma'}^\gamma(E) = \sum_{\chi=b,1} I_{\alpha\alpha'}^{\gamma,\chi}(E)\hat{L}_{\sigma\sigma'}^\chi$, where $\hat{L}_{\sigma\sigma'}^{a,b} = L^{a,b}\delta_{\sigma\sigma'}$ and $\hat{L}_{\sigma\sigma'}^i = \underline{L}^i\sigma_{\sigma\sigma'}$ for $i = 1, 2, 3$ (σ are the Pauli matrices). The superoperators $L^{a,b}$ and $\underline{L}^{1,2,3}$ can be expressed in terms of the spin superoperators $\underline{L}^+ = \underline{S}$ and $\underline{L}^- = \cdot(-\underline{S})$ via $L^a = \frac{3}{4} + \underline{L}^+\underline{L}^-$, $L^b = \frac{1}{4} - \underline{L}^+\underline{L}^-$, $\underline{L}^{1,3} = \frac{1}{2}(\underline{L}^+ - \underline{L}^- \mp 2i\underline{L}^+ \wedge \underline{L}^-)$, and $\underline{L}^2 = -\frac{1}{2}(\underline{L}^+ + \underline{L}^-)$. The closed algebra of these superoperators needed to evaluate the RG-equations can be found in Ref. 2. The algebra is applied after the summations over the creation/annihilation and spin indices η and σ have been performed. Furthermore, by using (11), the intermediate propagators are replaced by the c-numbers $\Pi(E + \bar{\omega}) \approx \frac{Z(E)}{\bar{\omega} + \chi(E)}$, where $Z(E) = \frac{1}{1+i\frac{\partial}{\partial E}(E)}$ and $\chi(E) = Z(E)(E + i\Gamma(E))$. This follows from the fact that the zero eigenvalue of the Liouvillian can be omitted, and the three nonzero eigenvalues are degenerate and given by $-i\Gamma(E)$.

A detailed analysis shows that the vertex component $I^{\gamma,b}$ is generated but does not influence the current kernel. Furthermore, with $G^2 \sim J$, it turns out that the vertex component G^a is generated in $O(J^2)$ and influences the Liouvillian only in $O(J^4)$. Therefore it can be omitted as well and we are left with a set of closed equations for the vertex components G^2 , G^3 and $I^{\gamma,1}$, together with the rates Γ and Γ_γ . For applied voltages the energy arguments are shifted by multiples of the chemical potentials of the reservoirs and all components couple to each other. After the summations over the indices η and σ have been performed, we use the shorthand notation $1 \equiv \alpha$ for the reservoir index. Furthermore, we define $\mu_{12} = \mu_{\alpha_1} - \mu_{\alpha_2}$, $E_{12} = E + \mu_{12}$ and $E_{1234} = E + \mu_{12} + \mu_{34}$. For convenience we define $J_{12}(E) = -G_{12}^2(E)$, $K_{12}(E) = -i\frac{2}{\pi}G_{12}^3(E_{21})$ and $I_{12}^\gamma(E) = -4I_{12}^{\gamma,1}(E)$. From hermiticity of the original Hamiltonian, we obtain the symmetry properties $J_{12}(E)^* = J_{21}(-E^*)$, $K_{12}(E)^* = K_{21}(-E^*)$ and $I_{12}^\gamma(E)^* = -I_{21}^\gamma(-E^*)$ for the vertex components, and $\Gamma(E)^* = \Gamma(-E^*)$ and $\Gamma_\gamma(E)^* = \Gamma_\gamma(-E^*)$ for the rates². The initial conditions at high energies arising from the symmetric part of the Fermi function are calculated in Ref. 2 as

$$Z = 1 \quad , \quad K_{\alpha\alpha'} = \sum_{\alpha''} J_{\alpha\alpha''} J_{\alpha''\alpha'} \quad , \quad I_{\alpha\alpha'}^\gamma = (\delta_{\alpha\gamma} - \delta_{\alpha'\gamma}) J_{\alpha\alpha'} \quad , \quad \pi\Gamma_\gamma = \frac{3\pi^2}{4} \sum_{\alpha} \mu_{\gamma\alpha} J_{\gamma\alpha} J_{\alpha\gamma} \quad , \quad (15)$$

where $J_{\alpha\alpha'} = J_{\alpha'\alpha}^*$ are the initial exchange couplings in the interaction part $H_{ex} = \sum_{\alpha\alpha'} J_{\alpha\alpha'} \underline{s}_{\alpha\alpha'} \underline{S}$ of the Hamiltonian, where $\underline{s}_{\alpha\alpha'} = \frac{1}{2} \sum_{\sigma k, \sigma' k'} a_{\alpha\sigma k}^\dagger \sigma_{\sigma\sigma'} a_{\alpha'\sigma' k}$.

The full RG equations at finite voltages and temperature are quite involved and will be discussed in detail in a forthcoming work⁵. Here we consider the equations either for $\mu_\alpha = 0$ or for $T = 0$, which simplifies them considerably. Furthermore, we have taken the special case of two reservoirs with $\mu_L = -\mu_R = \frac{V}{2}$ and used the special initial condition $J_{\alpha\alpha'} = 2\sqrt{x_\alpha x_{\alpha'}} J$ with $0 < x_\alpha < 1$ and $x_L + x_R = 1$ (called “2-reservoir case” in the following). In this case the conductance in units of $G_0 = \frac{2e^2}{h}$ follows from $G = G(E = 0)$ with $G(E) = \pi \frac{\partial}{\partial V} \Gamma_L(E)$. In the general case, the variation of the current flowing in reservoir γ is given in Laplace space by $\delta I_\gamma(E) = \frac{2e}{h} \frac{i}{E} \pi \delta \Gamma_\gamma(E)$. The spin dynamics follows from $\langle \underline{S} \rangle(E) = \frac{i}{E + i\Gamma(E)} \langle \underline{S} \rangle(t = 0)$.

For zero temperature $T = 0$ but arbitrary nonzero μ_α a lengthy but straightforward calculation gives the following RG equations

$$\frac{\partial^2}{\partial E^2} \Gamma(E) = i\Pi_{12} J_{12}(E) J_{21}(E_{12}) \quad , \quad \frac{\partial}{\partial E} \delta \Gamma(E) = i\delta\mu_{12} \Pi_{12} J_{12}(E) J_{21}(E_{12}), \quad (16)$$

$$\frac{\partial}{\partial E} (\pi \delta \Gamma_\gamma(E)) = -\frac{3\pi^2}{4} \Pi_{12} I_{12}^\gamma(E) K_{21}(E) (\delta\mu_{12} + iZ_{12} \delta \Gamma_{12}), \quad (17)$$

$$\begin{aligned} \frac{\partial}{\partial E} J_{12}(E) = & -\frac{1}{2} \Pi_{13} J_{13}(E) J_{32}(E_{13}) - \frac{1}{2} \Pi_{32} J_{13}(E_{32}) J_{32}(E) \\ & - \frac{1}{4} J_{12}(E_{34}) J_{34}(E) J_{43}(E_{1234}) (Z_{34} \Pi_{1234} + Z_{1234} \Pi_{34}), \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{\partial}{\partial E} K_{12}(E) = & -\Pi_{23} J_{13}(E_{21}) K_{32}(E) - \Pi_{31} K_{13}(E) J_{32}(E_{21}) \\ & + \frac{1}{2} J_{12}(E_{2134}) J_{34}(E_{21}) K_{43}(E) (Z_{34} \Pi_{2134} + Z_{2134} \Pi_{34}), \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{\partial}{\partial E} I_{12}^\gamma(E) = & -\Pi_{13} I_{13}^\gamma(E) J_{32}(E_{13}) - \Pi_{32} J_{13}(E_{32}) I_{32}^\gamma(E) \\ & + \frac{1}{2} J_{12}(E_{34}) I_{34}^\gamma(E) J_{43}(E_{1234}) (Z_{34} \Pi_{1234} + Z_{1234} \Pi_{34}), \end{aligned} \quad (20)$$

where $Z_{12} = Z(E_{12})$, $Z(E) = \frac{1}{1+i\frac{d}{dE}}$, $\delta\Gamma_{12} = \delta\Gamma(E_{12})$, $\Pi_{12} = \Pi(E_{12})$ and $\Pi_{1234} = \Pi(E_{1234})$, with $\Pi(E) = \frac{1}{E+i\Gamma(E)}$. These equations are even valid for an arbitrary number of reservoirs and arbitrary initial matrix $J_{\alpha\alpha'}$. We emphasize again that in these equations the index $1 \equiv \alpha_1$ contains only the reservoir label, and the new notations $E_{12} = E + \mu_{12}$, $\mu_{12} = \mu_1 - \mu_2$, and $E_{1234} = E + \mu_{12} + \mu_{34}$ have been introduced.

For zero voltages $\mu_\alpha = 0$ we consider the “2-reservoir case” and parameterize the renormalized vertices $J_{12}(E) = 2\sqrt{x_1 x_2} J(E)$, $K_{12}(E) = 2\sqrt{x_1 x_2} K(E)$ and $I_{12}^\gamma(E) = (\delta_{\alpha_1 \gamma} - \delta_{\alpha_2 \gamma}) 2\sqrt{x_1 x_2} J_I(E)$. Omitting the common E -argument of all quantities, we obtain the following RG equations

$$\frac{\partial^2 \Gamma}{\partial E^2} = 4iJ^2(F^{(1)} - 4JF^{(2)}), \quad \frac{\partial J}{\partial E} = -2J^2(F^{(3)} + ZJF^{(1)} - 2JF^{(4)}), \quad (21)$$

$$\frac{\partial K}{\partial E} = -4KJ(F^{(3)} - ZJF^{(1)} + 2JF^{(4)}), \quad \frac{\partial J_I}{\partial E} = -2J_I JF^{(3)}, \quad (22)$$

$$\frac{\partial G}{\partial E} = -6\pi^2 x_L x_R J_I K(F^{(1)} - 6JF^{(2)}), \quad (23)$$

where we have defined the integrals

$$F^{(1)}(E) = Z(E) \int d\omega \int d\omega' \frac{f'(\omega)f'(\omega')}{\omega + \omega' + \chi(E)} = Z(E) \sum_{n,m=0}^{\infty} \frac{2(2\pi iT)^2}{(i\omega_n + i\omega_m + \chi(E))^3} = \frac{Z(E)}{2\pi iT} \frac{d^2}{d\rho^2} [\rho\psi(\rho)], \quad (24)$$

$$F^{(2)}(E) = Z(E) \int d\omega \int d\omega' \frac{\mathcal{F}(E, \omega)f'(\omega)f'(\omega')}{\omega + \omega' + \chi(E)} \\ = -\frac{Z^2(E)}{2\pi iT} \left\{ [1 + \psi(\rho + \frac{1}{2})] \frac{d^2}{d\rho^2} [\rho\psi(\rho)] - \frac{1}{2} \frac{d^2}{d\rho^2} [\rho\psi(\rho)\psi(\rho + 1)] \right\}, \quad (25)$$

$$F^{(3)}(E) = -Z(E) \int d\omega \frac{f'(\omega)}{\omega + \chi(E)} = Z(E) \sum_{n=0}^{\infty} \frac{2\pi iT}{(i\omega_n + \chi(E))^2} = \frac{Z(E)}{2\pi iT} \frac{d}{d\rho} \psi(\rho + \frac{1}{2}), \quad (26)$$

$$F^{(4)}(E) = Z(E) \int d\omega \frac{\mathcal{F}(E, \omega)f'(\omega)}{\omega + \chi(E)} = \frac{Z^2(E)}{2\pi iT} \left\{ \frac{1}{2} \frac{d}{d\rho} [\psi^2(\rho + \frac{1}{2})] + \sum_{n=0}^{\infty} \frac{d}{d\rho} \left[\frac{\psi(n + 1 + \rho)}{n + \frac{1}{2} + \rho} \right] \right\}, \quad (27)$$

and

$$\mathcal{F}(E, \omega) = Z(E) \int d\omega' f^a(\omega') \left(\frac{1}{\omega + \omega' + \chi(E)} - \frac{1}{\omega' + \chi(E)} \right) \\ = -Z(E) 2\pi iT \sum_{n=0}^{\infty} \left(\frac{1}{\omega + i\omega_n + \chi(E)} - \frac{1}{i\omega_n + \chi(E)} \right), \quad (28)$$

with $f^a(\omega) = f(\omega) - \frac{1}{2} = -T \sum_n \frac{1}{\omega - i\omega_n}$ and the Matsubara frequencies $\omega_n = 2\pi T(n + \frac{1}{2})$. As seen above, all these integrals can be expressed through the digamma function ψ of the argument $\rho = \frac{\chi(E)}{2\pi iT}$.

In the following sections we will analyze these RG equations analytically in certain regimes. Furthermore, we will explain how to set up the initial conditions in order to enable an achievement of the scaling limit.

A. $\mathbf{T=V=0}$

At $T = V = 0$ we use the RG equations (21)-(23) together with $F^{(1)} = F^{(3)} = \frac{1}{E+i\Gamma(E)}$ and $F^{(2)} = F^{(4)} = 0$. With $E = i\Lambda$ and $\lambda = \Lambda + \Gamma(i\Lambda)$ we obtain

$$d_\Lambda^2 \Gamma = -\frac{4J^2}{\lambda}, \quad d_\Lambda J = -\frac{2J^2(1 + ZJ)}{\lambda}, \quad d_\Lambda K = -\frac{4KJ(1 - ZJ)}{\lambda}, \quad d_\Lambda J_I = -\frac{2J_I J}{\lambda}, \quad d_\Lambda G = -4x_L x_R \frac{3\pi^2}{2} \frac{J_I K}{\lambda}, \quad (29)$$

where $Z = \frac{1}{1+d_\Lambda \Gamma}$. With $\tilde{J} = ZJ$, $\tilde{J}_I = ZJ_I$ and $l = \ln \frac{\lambda_0}{\lambda}$ the equations can be written as

$$\frac{dZ}{dl} = -4Z\tilde{J}^2, \quad \frac{d\tilde{J}}{dl} = 2\tilde{J}^2(1 - \tilde{J}), \quad \frac{dK}{dl} = 4K\tilde{J}(1 - \tilde{J}), \quad \frac{d\tilde{J}_I}{dl} = 2\tilde{J}_I\tilde{J}(1 - 2\tilde{J}), \quad \frac{dG}{dl} = 4x_L x_R \frac{3\pi^2}{2} \tilde{J}_I K. \quad (30)$$

These equations have the following invariants which are fixed as follows

$$T_K = \lambda e^{-\frac{1}{2\tilde{J}}} \frac{\tilde{J}^{1/2}}{(1-\tilde{J})^{1/2}}, \quad 1 = \frac{Z}{(1-\tilde{J})^2}, \quad 2 = \frac{K}{\tilde{J}^2}, \quad 1 = \frac{\tilde{J}_I}{\tilde{J}(1-\tilde{J})}. \quad (31)$$

Hereby T_K sets the energy scale and defines the Kondo temperature. The other invariants are fixed by comparison with the initial conditions (15) in the scaling limit $\Lambda \rightarrow \infty$ and $\tilde{J} \rightarrow 0$ such that T_K is kept constant. The initial conditions (15) are given by $Z = 1$, $K = 2J^2$, and $J_I = J$, which, in the scaling limit, lead to the invariants (31). Using the results for \tilde{J}_I and K , the RG equation for the conductance can be integrated to

$$G = 4x_L x_R \frac{3\pi^2}{4} \tilde{J}^2, \quad (32)$$

where the integration constant has been fixed again by comparison to the initial condition (15). Having fixed the invariants all information about the microscopic details of the model parameters is swept into the single energy scale T_K . The only remaining differential equation to be solved numerically is $d_\Lambda \lambda = \frac{1}{Z} = \frac{1}{(1-\tilde{J})^2}$, where \tilde{J} is a function of λ via the expression for T_K . To solve it we need the initial value $\lambda(0) = \Gamma(0) \equiv \bar{\Gamma}$ or, equivalently, the initial value $\tilde{J}(0) \equiv \bar{J}$. This value is determined from the condition that the normalized conductance is unitary $G(0)/(4x_L x_R) = 1$ at $T = V = 0$ for the 1-channel Kondo model. This yields the values $\bar{J} = \frac{2}{\pi\sqrt{3}} \approx 0.37$ and $\bar{\Gamma} \approx 5.11 T_K$. From the numerical solution we obtain all quantities at some high energy scale $\Lambda_0 \gg T_K$. At finite T or μ_α , we take these values as initial condition at $\Lambda_0 \gg T, |\mu_\alpha|, T_K$ and solve the RG equations starting from Λ_0 .

In second order truncation we neglect the terms $\sim J^3, KJ^2$ on the r.h.s. of (29) and obtain $T_K = \lambda J e^{-\frac{1}{2J}}$, $Z = \frac{1}{1+2J}$, $K = 2J^2$, $J_I = J$ and $G = 4x_L x_R \frac{3\pi^2}{4} J^2$. Fixing unitary conductance yields $\bar{J} = J(0) = \frac{2}{\pi\sqrt{3}} \approx 0.37$ and $\bar{\Gamma} = \Gamma(0) = \frac{1}{J} e^{\frac{1}{2J}} \approx 10.57 T_K$ for the initial values for Γ and J at $\Lambda = 0$.

Weak coupling.—To obtain the spin relaxation rate $\Gamma(i\Lambda)$ in the weak coupling regime $\Lambda \gg T_K$, we expand \tilde{J} and Γ around the 2-loop poor man scaling solution $J_p(\Lambda)$, defined by $\ln \frac{\Lambda}{T_K} = \frac{1}{2J_p} + \frac{1}{2} \ln \frac{1-J_p}{J_p}$. Using $\Gamma \sim \Lambda O(J_p)$ and $\tilde{J} = J_p + O(J_p^2)$ we find $\lambda = \Lambda(1 + O(J_p))$ and $\frac{1}{2J} = \ln \frac{\lambda}{T_K} - \frac{1}{2} \ln \frac{1-\tilde{J}}{\tilde{J}} = \frac{1}{2J_p} + O(J_p)$. This can only be fulfilled if $\tilde{J} = J_p + O(J_p^3)$, from which we obtain $d_\Lambda \lambda = \frac{1}{Z} = \frac{1}{(1-\tilde{J})^2} = 1 + 2J_p + 3J_p^2 + O(J_p^3)$. Using the ansatz $\lambda = \Lambda(1 + aJ_p + bJ_p^2 \ln J_p + cJ_p^2 + O(J_p^3))$ and $d_\Lambda J_p = -2J_p^2(1-J_p)/\Lambda$, we find $d_\Lambda \lambda = 1 + aJ_p + bJ_p^2 \ln J_p + (c-2a)J_p^2 + O(J_p^3)$. Comparing the two expressions yields $a = 2$, $b = 0$ and $c = 2a + 3 = 7$, i.e. with $E = i\Lambda$

$$\tilde{J}(E) = J_p(-iE) + O(J_p^3), \quad \Gamma(E) = -2iE \left\{ J_p(-iE) + \frac{7}{2} J_p^2(-iE) + O(J_p^3) \right\}. \quad (33)$$

Time evolution.—The spin dynamics is calculated from the inverse Laplace transform $\langle \underline{S} \rangle(t) = \frac{1}{2\pi} \int dE \frac{e^{-iEt}}{\lambda(E)} \langle \underline{S} \rangle(0)$, with $\lambda(E) = -iE + \Gamma(E)$, where the integration contour lies slightly above the real axis. The resolvent has a pole at $\lambda(E) = 0$, which corresponds to the fixed point $\tilde{J}(E) = J^* = 1$ since $\lambda = T_K e^{-\frac{1}{2\tilde{J}}} (1-\tilde{J})^{1/2} / \tilde{J}^{1/2}$. The fixed point is reached for $-iE = \Lambda^* = -\Gamma^* < 0$ since at $\Lambda = 0$ we have chosen $\tilde{J}(0) = \frac{2}{\pi\sqrt{3}} < 1$. Close to the fixed point $\Lambda \sim \Lambda^* = -\Gamma^*$ we get $\frac{dZ}{d\Lambda} = 4Z\tilde{J}^2/\lambda \approx \alpha Z/\lambda$ with $\alpha = 4J^{*2}$. This leads to $Z \sim \lambda^\alpha$. Using $\frac{d\lambda}{d\Lambda} = \frac{1}{Z} \sim \lambda^{-\alpha}$ we get $\lambda \sim (\Lambda - \Lambda^*)^{\frac{1}{1+\alpha}}$. This means that the resolvent $\frac{1}{\lambda(E)}$ has a branching pole at $E = -i\Gamma^*$ followed by a branch cut on the line $E = -i(\Gamma^* + x)$ with $0 < x < \infty$. The contour integral can be closed in the lower half around the branch cut and, in the long-time limit $t \gg \frac{1}{\Gamma^*}$, the spin dynamics follows from

$$\langle \underline{S} \rangle(t) = \frac{1}{\pi} e^{-\Gamma^* t} \int_0^\infty dx e^{-xt} \text{Im} \frac{1}{\lambda(i(\Lambda^* - x - i\eta))} \langle \underline{S} \rangle(0) \sim e^{-\Gamma^* t} \int_0^\infty dx e^{-xt} \frac{1}{x^{\frac{1}{1+\alpha}}} \sim \frac{1}{t^g} e^{-\Gamma^* t}, \quad (34)$$

with the exponent

$$g = \frac{\alpha}{1+\alpha} = \frac{4J^{*2}}{1+4J^{*2}}. \quad (35)$$

As a result the exponent depends on the fixed point value J^* , which in turn depends on the truncation order. In third order truncation we get $J^* = 1$ leading to $g = \frac{4}{5}$. In second order truncation we neglect the term of $O(J^3)$ in the RG equation (29) for J . This yields $\frac{d\tilde{J}}{d\Lambda} = 2\tilde{J}^2(1-2\tilde{J})$ and $Z = 1-2\tilde{J}$. The fixed point in second order truncation is at $\tilde{J} = J^* = \frac{1}{2}$ which yields $g = \frac{1}{2}$. A fixed point value $J^* = \infty$ would lead to $g = 1$. This can be

achieved by changing the β -function by hand to $\frac{d\tilde{J}}{d\lambda} = \frac{2\tilde{J}^2}{1+\tilde{J}}$, which is still consistent with the β -function in third order truncation. This leads to the invariant $T_K = \lambda\tilde{J}^{1/2}e^{-\frac{1}{2\tilde{J}}}$ and to $Z = e^{-2\tilde{J}-\tilde{J}^2}$. Close to the fixed point $\lambda = 0$, the coupling $\tilde{J} \sim \frac{1}{\lambda^2}$ tends to infinity and fulfills the differential equation $d_\Lambda \tilde{J} = -\frac{1}{2\lambda} \frac{2\tilde{J}^2}{1+\tilde{J}} \sim -\tilde{J}^{3/2}e^{2\tilde{J}+\tilde{J}^2}$. In the leading order we get $\tilde{J} \sim (-\ln \frac{\Lambda-\Lambda^*}{T_K})^{1/2}$ and $\lambda \sim \tilde{J}^{-1/2} \sim (-\ln \frac{\Lambda-\Lambda^*}{T_K})^{-1/4}$, which results in a pre-exponential function $\sim \frac{1}{t(\ln T_K t)^{3/4}}$. Thus, besides the power-law $\frac{1}{t}$ the pre-exponential function contains an additional logarithmic term in this case. In conclusion, the pre-exponential modulation depends crucially on the form of the β -function and can not be determined within a perturbative truncation scheme. Instead, one needs a non-perturbative analysis of the β -function close to the fixed point which goes beyond the scope of this work.

The problem of obtaining the correct pre-exponential modulation for the 1-channel Kondo model is analogous to the problem of obtaining the correct scaling dimension close to the fixed point for multi-channel Kondo models. For channel number $N > 1$ the RG equations for Z and \tilde{J} change to $\frac{dZ}{d\lambda} = -4NZ\tilde{J}^2$ and $\frac{d\tilde{J}}{d\lambda} = 2\tilde{J}^2(1-N\tilde{J})$, giving $T_K = \lambda e^{-\frac{1}{2\tilde{J}}}(\frac{\tilde{J}}{1-N\tilde{J}})^{N/2}$ and $Z = (1-N\tilde{J})^2$. The fixed point $\lambda = 0$ corresponds to $\tilde{J} = \frac{1}{N}$ and the system should reveal non-Fermi liquid behavior at low energies^{6,7}. This means that we have to choose $\tilde{J}(E=0) = \frac{1}{N}$ identical to the fixed point value such that the pole $\lambda = 0$ lies at the origin $E = \Lambda^* = 0$ of the complex plane with $\tilde{\Gamma}(0) = \Gamma^* = 0$. In contrast to the exponential decay for the 1-channel case, this leads to a pure power-law decay in the long-time limit, which is another signature of non-Fermi liquid behavior. Expanding around the fixed point we get $\frac{dZ}{d\lambda} \approx \alpha Z/\lambda$ with $\alpha = 4NJ^{*2}$. According to (35) this gives the power-law exponent $g = \frac{\alpha}{1+\alpha} = \frac{4NJ^{*2}}{1+4NJ^{*2}}$. Inserting $J^* = \frac{1}{N}$ we obtain $g = \frac{4}{N+4}$. However, this result is only consistent for $N \gg 1$, where the β -function can be systematically truncated and the system stays in the weak-coupling limit⁸. For small N the result is not reliable like in the 1-channel case discussed above.

B. T=0 and finite voltage

At zero temperature we start from the RG equations (16-20) and discuss analytically the “2-reservoir case” in the weak coupling regime of large voltage $V \gg T_K$ and the strong coupling regime of small voltage $V \ll T_K$.

Weak coupling.—To obtain the weak coupling expansion for $V \gg T_K$ we follow Ref. 3 and discuss the solution of the RG equations separately for $\Lambda > V$ and $\Lambda < V$, where $\Lambda = -iE$. For $\Lambda > V$, we expand the solution of the RG equations around a reference solution, where all propagators are replaced by $\frac{Z}{\Lambda}$. No voltage dependence is generated in this case and the RG equations obtain the same form as in (30) but with $l \rightarrow \ln \frac{\Lambda_0}{\Lambda}$. As a consequence, the reference solution is given by (31) but with $\tilde{J} \rightarrow J_p$, where $J_p(\Lambda)$ is the 2-loop poor man scaling solution, already introduced in section II A. To calculate the first correction to the reference solution for $\Lambda > V$, we insert the exact identity

$$\begin{aligned} \frac{1}{\Lambda_{12} + \Gamma_{12}} H_\Lambda &= \frac{Z}{\Lambda} H_\Lambda + \frac{\partial}{\partial \Lambda} \left\{ Z_{12} \left(\ln \frac{\Lambda_{12} + \Gamma_{12}}{\Lambda} \right) H_\Lambda \right\} \\ &\quad - \left(\ln \frac{\Lambda_{12} + \Gamma_{12}}{\Lambda} \right) \frac{\partial}{\partial \Lambda} \{ Z_{12} H_\Lambda \} + \frac{Z_{12} - Z}{\Lambda} H_\Lambda \end{aligned} \quad (36)$$

for all propagators occurring in the RG equations (16-20), where $\Lambda_{12} = \Lambda - i\mu_{12}$ and H_Λ stands symbolically for the rest of some term on the r.h.s. of the RG equation involving products of various couplings. We get $H_\Lambda \sim J_p^2$ ($H_\Lambda \sim J_p^3$) for $d_\Lambda J, d_\Lambda I^\gamma$ and $d_\Lambda Z$ ($d_\Lambda K$ and $d_\Lambda G$, with $G = \pi \frac{\partial}{\partial V} \Gamma_L$). The first term of (36) generates the reference solution, which, according to (31) and (32), gives $O(J_p)$ ($O(J_p^2)$) for J, I^γ and $Z - 1$ (K and G). We are aiming at calculating all quantities one order beyond the reference solution. To estimate the various terms of (36) we use $\ln \frac{\Lambda_{12} + \Gamma_{12}}{\Lambda} \sim O(\frac{V}{\Lambda}, \frac{\Gamma}{\Lambda}) \sim O(\frac{V}{\Lambda}, J_p)$ for $\Lambda \gg V, \Gamma$, where we used $\Gamma \sim O(\Lambda J_p)$ according to (33). As a consequence, the third term of (36) is $\sim O(\frac{V}{\Lambda^2} J_p^3, \frac{1}{\Lambda} J_p^4)$ ($\sim O(\frac{V}{\Lambda^2} J_p^4, \frac{1}{\Lambda} J_p^5)$) for $d_\Lambda J, d_\Lambda I^\gamma$ and $d_\Lambda Z$ ($d_\Lambda K$ and $d_\Lambda G$). Using $\frac{\partial Z}{\partial \Lambda} \sim \frac{1}{\Lambda} O(J_p^2)$, the fourth term of (36) is $\sim O(\frac{V}{\Lambda^2} J_p^4)$ ($\sim O(\frac{V}{\Lambda^2} J_p^5)$). Thus, after integrating over Λ , the third and fourth terms of (36) produce corrections $\sim J_p^3$ ($\sim J_p^4$) for J, I^γ and Z (K and G), i.e. two orders higher than the reference solution. In contrast, the second term of (36) produces the first correction $\sim \frac{V}{\Lambda} O(J_p^2)$ ($\sim \frac{V}{\Lambda} O(J_p^3)$) to the reference solution. We can neglect the influence of this correction on the first term of (36), since this gives contributions of the same order as the third term of (36). Thus for our purpose it is sufficient to take into account the first two terms of (36) and calculate H_Λ by using the reference solution. For the RG equation (17) for the conductance this means that the term involving $\delta\Gamma_{12} \sim \delta V J_p$ can be neglected on the r.h.s. and the first term on the r.h.s. leads to the first correction to

the reference solution, i.e. the weak coupling expansion in the regime $\Lambda > V$ reads

$$G(E = i\Lambda, V) = 4x_L x_R \frac{3\pi^2}{4} J_p^2(\Lambda) \left\{ 1 - 2J_p(\Lambda) \sum_{\alpha_1 \neq \alpha_2} \ln \frac{\Lambda_{12} + i\Gamma_{12}}{\Lambda} + O(J_p^2) \right\}. \quad (37)$$

Following Ref. 3 we use this result at $\Lambda = V$ as initial condition to solve the RG equation for the conductance in the following regime $\Lambda < V$ by expanding in the reference solution $J_p(V)$ evaluated at $\Lambda = V$. Replacing all vertices on the r.h.s. of the RG equation for the conductance by the reference solution at $\Lambda = V$, we get with $H_V \sim J_I K \sim O(J_p^3)$ and $\frac{\partial Z}{\partial \Lambda} \sim \frac{1}{\Lambda + \Gamma} O(J_p^2)$ instead of (36) the expression $\frac{1}{\Lambda_{12} + \Gamma_{12}} H_V = \frac{\partial}{\partial \Lambda} \{ Z_{12} (\ln \frac{\Lambda_{12} + \Gamma_{12}}{V}) H_V \} + \frac{1}{\Lambda + \Gamma} O(J_p^5)$. Applying $\int_V^0 d\Lambda$ to this expression and adding the initial condition (37) evaluated at $\Lambda = V$, the second term on the r.h.s. of (37) obviously cancels against the contribution from the lower bound $\Lambda = V$ of the integration. The remaining first term of (37) evaluated at $\Lambda = V$ together with the contribution from the upper bound $\Lambda = 0$ of the integration gives for the conductance at $E = 0$ the result $G = 4x_L x_R \frac{3\pi^2}{4} J_p^2(V) \left\{ 1 - 2J_p(V) \sum_{\alpha_1 \neq \alpha_2} \ln \frac{-i\mu_{12} + i\Gamma(\mu_{12})}{V} + O(J_p^2) \right\}$, where we used $Z_{12}|_{\Lambda=0} = 1 + O(J_p(V))$. With $\mu_{12} = \alpha_{12} \frac{V}{2}$ and $\alpha_{12} = \alpha_1 - \alpha_2$, we find

$$\sum_{\alpha_1 \neq \alpha_2} \ln \frac{-i\mu_{12} + i\Gamma(\mu_{12})}{V} = \sum_{\alpha_1 \neq \alpha_2} \ln(-i\frac{1}{2}\alpha_{12}) + O(\frac{\Gamma}{V}) = \ln(-i) + \ln(i) + O(\frac{\Gamma}{V}) = O(\frac{\Gamma}{V}) \sim J_p^2, \quad (38)$$

where we have used $\Gamma(E = 0) \sim O(V J_p^2)$ which can be derived from an analogous weak coupling expansion. Thus, we can neglect this term and finally get the weak coupling result

$$G(V) = 4x_L x_R \frac{3\pi^2}{4} J_p^2(V) + O(J_p^4). \quad (39)$$

Strong coupling.—In the strong coupling regime $V \ll T_K$ the 1-channel Kondo model behaves like a local Fermi liquid⁶, which, in particular, implies a quadratic dependence of the differential conductance on the applied bias voltage^{9,10} $G = 4x_L x_R \{1 - c_V (V/\bar{\Gamma})^2 + O((V/\bar{\Gamma})^3)\}$. As a reference scale we use here $\bar{\Gamma} = \Gamma(E = 0)$. In the following we recover it by means of the RTRG as well as provide a quantitative value for the Fermi-liquid coefficient c_V up to the next-to-leading order in the renormalized coupling constant $\tilde{J} = \tilde{J}(E = 0)|_{T=V=0}$. This expansion can be performed by differentiating the RG equation (17) for the normalized conductance twice by the voltage and integrate it over E . A more direct path is to consider the original diagrammatic series and calculate directly the third order variation $\delta^3 \Sigma_\gamma(E) = \frac{1}{3!} \frac{\partial^3}{\partial V^3} \Sigma_\gamma(E) \delta V^3$ of the current kernel by using the diagrammatic method explained in Section I. Of particular advantage is the fact that the variation $\delta \Gamma(E)$ of the Liouvillian vanishes at $V = T = 0$ for any E . This follows directly from the RG equation (16) after inserting the algebra of the vertices at $V = T = 0$ derived in Section II A, given by

$$\begin{aligned} J_{\alpha_1 \alpha_2} &= 2\sqrt{x_{\alpha_1} x_{\alpha_2}} J, & K_{\alpha_1 \alpha_2} &= 2\sqrt{x_{\alpha_1} x_{\alpha_2}} K, & I_{\alpha_1 \alpha_2}^\gamma &= (\delta_{\alpha_1 \gamma} - \delta_{\alpha_2 \gamma}) 2\sqrt{x_{\alpha_1} x_{\alpha_2}} J_I, \\ Z &= (1 - \tilde{J})^2, & K &= 2\tilde{J}^2, & \tilde{J}_I &= \tilde{J}(1 - \tilde{J}), \end{aligned} \quad (40)$$

where $\tilde{J} = ZJ$ and $\tilde{J}_I = ZJ_I$. As a consequence we can express the variation of the propagator at $V = 0$ by derivatives with respect to the Laplace variable $\delta^n \Pi_{12} = \frac{1}{n!} (\delta \bar{\mu}_{12})^n \partial_E^n \Pi_{12}$. With this property we get for the third order variation of the current kernel at zero voltage

$$\begin{aligned} \delta^3 \Sigma_\gamma(E)|_{V=0} &= \frac{1}{2} \frac{1}{3!} (\delta \bar{\mu}_{12})^3 \text{Diagram 1} \\ &+ \frac{1}{2} (\delta \bar{\mu}_{12})^2 \delta \bar{\mu}_{13} \text{Diagram 2} + \frac{1}{2} \delta \bar{\mu}_{12} (\delta \bar{\mu}_{13})^2 \text{Diagram 3} + O(\tilde{J}^5), \end{aligned} \quad (41)$$

where all diagrams have to be evaluated at $V = 0$ on the r.h.s. The corrections are of $O(\tilde{J}^5)$ since for the current kernel one of the vertices has to be $K \sim \tilde{J}^2$. At $V = 0$ the last two diagrams are identical and, by using partial integration, can be written as

$$\text{Diagram 2} = \text{Diagram 3} = -\frac{1}{2} \text{Diagram 4} \quad (42)$$

Analogously to (5) we can perform two partial integrations with the first term on the r.h.s. of (41) and, by again using (42), we arrive at

$$\begin{aligned} \frac{\partial^3}{\partial V^3} \Sigma_\gamma(E)|_{V=0} \delta V^3 &= \frac{1}{2} (\delta \bar{\mu}_{12})^3 \text{ (diagram with 3 vertices and 3 internal lines)} \\ &+ \frac{1}{2} \{ (\delta \bar{\mu}_{12})^3 + (\delta \bar{\mu}_{13})^3 - 3(\delta \bar{\mu}_{12})^2 \delta \bar{\mu}_{13} - 3\delta \bar{\mu}_{12} (\delta \bar{\mu}_{13})^2 \} \text{ (diagram with 4 vertices and 4 internal lines)} + O(\tilde{J}^5). \end{aligned} \quad (43)$$

Due to the derivative of the Fermi functions in all contractions, all frequencies are set to zero and the diagrams can be straightforwardly evaluated by performing the sums over the indices η and σ by using the algebra of the vertices together with their value (40) at $V = 0$. With $\mu_\alpha = \frac{1}{2}\alpha V$, $\alpha_{12} = \alpha_1 - \alpha_2$ and $\delta_{\alpha_1\gamma} - \delta_{\alpha_2\gamma} = \frac{1}{2}\gamma\alpha_{12}$, we obtain after a straightforward calculation

$$\begin{aligned} \frac{\partial^3}{\partial V^3} \Sigma_\gamma(E)|_{V=0} \delta V^3 &= i\gamma L^b 4x_L x_R \frac{\pi}{64} \frac{\tilde{J}_I K(1+3\tilde{J})}{Z^2(E+i\Gamma)^2} \\ &\times \left\{ \sum_{\alpha_1\alpha_2} (\alpha_{12})^4 + \sum_{\alpha_1\alpha_2\alpha_3} \alpha_{12} ((\alpha_{12})^3 + (\alpha_{13})^3 - 3(\alpha_{12})^2\alpha_{13} - 3\alpha_{12}(\alpha_{13})^2) \right\} \delta V^3 + O(\tilde{J}^5). \end{aligned} \quad (44)$$

Performing the sums over α_i and using $\Sigma_\gamma(E) = i\Gamma_\gamma(E)L^b$ along with $G = \pi \frac{\partial}{\partial V} \Gamma_L$ and (40), we obtain at $E = 0$ the final result

$$c_V = -\frac{1}{2} \frac{\bar{\Gamma}^2}{4x_L x_R} \frac{\partial^3}{\partial V^3} \pi \Gamma_\gamma(E=0) = \frac{3\pi^2}{2} \frac{\bar{J}^3(1+3\bar{J})}{(1-\bar{J})^3} + O(\tilde{J}^5). \quad (45)$$

Finally, we note that the second variation of the current kernel at $V = 0$ yields zero since an analogous derivation as above gives

$$\begin{aligned} \delta^2 \Sigma_\gamma(E)|_{V=0} &= \frac{1}{2} \frac{1}{2} (\delta \bar{\mu}_{12})^2 \text{ (diagram with 3 vertices and 2 internal lines)} + \delta \bar{\mu}_{12} \delta \bar{\mu}_{13} \text{ (diagram with 4 vertices and 3 internal lines)} + O(\tilde{J}^5), \\ &= \frac{1}{4} (\delta \bar{\mu}_{12})^2 \text{ (diagram with 3 vertices and 2 internal lines)} - \frac{1}{2} (\delta \bar{\mu}_{12} - \delta \bar{\mu}_{13})^2 \text{ (diagram with 4 vertices and 3 internal lines)} + O(\tilde{J}^5). \end{aligned} \quad (46)$$

Inserting the algebra of the vertices along with their values (40) at $V = 0$, and using $\sum_{\alpha_1\alpha_2} (\alpha_{12})^3 = 0$ and $\sum_{\alpha_1\alpha_2\alpha_3} (\alpha_{12} - \alpha_{13})^2 \alpha_{12} = 0$, we find a vanishing value. This shows that the corrections of the conductance to the unitary value start with a term $\sim V^2$ as required by the Fermi liquid theory.

C. $V=0$ and finite temperature

At zero voltage we consider the "2-reservoir case" and start from the RG equations (21-23) to discuss analytically the weak and strong coupling regimes $T \gg T_K$ and $T \ll T_K$.

Weak coupling.—To obtain the weak coupling expansion for $T \gg T_K$ we proceed analogously to the derivation presented in Section II B. We set $E = i\Lambda$ and start with the regime $\Lambda > T$, where we expand in the reference solution $J_p(\Lambda)$. As already discussed in Sections II B and II A, the reference solution (31) with $\tilde{J} \rightarrow J_p$ is obtained if one replaces the integrals $F^{(i)}(i\Lambda)$ defined in (24-27) by $iF^{(1,3)} \rightarrow \frac{Z}{\Lambda}$ and $F^{2,4} \rightarrow 0$ in the RG equations (21-23). Therefore, we construct an analog of (36) by writing $iF^{(1,3)} = \frac{Z}{\Lambda} + Zd_\Lambda \tilde{F}^{(1,3)} + O(J_p^2/\Lambda)$, such that $\tilde{F}^{(1,3)} \sim O((\frac{T}{\Lambda})^2, J_p)$ for $\Lambda \gg T$. Using $\rho = \frac{Z(\Lambda+\Gamma)}{2\pi T}$ and $\frac{d\rho}{d\Lambda} = \frac{1}{2\pi T}(1 + O(J_p^2))$ together with (24) and (26), we find that this can be achieved by defining $\tilde{F}^{(1)} = \frac{d}{d\rho} \{\rho \Psi(\rho)\} - 1 - \ln \frac{Z\Lambda}{2\pi T}$ and $\tilde{F}^{(3)} = \Psi(\rho + \frac{1}{2}) - \ln \frac{Z\Lambda}{2\pi T}$. Furthermore, we write $iF^{(2,4)} = Zd_\Lambda \tilde{F}^{(2,4)} + O(J_p^2/\Lambda)$ such that $\tilde{F}^{(2,4)} \sim O(\frac{T}{\Lambda})^2$, which follows directly from the asymptotic expansion $F^{(2,4)} \sim \frac{T^2}{\Lambda^3}$ for $\Lambda \gg T$. The analog of (36) then reads

$$iF_\Lambda^{(i)} H_\Lambda = c_i \frac{Z_\Lambda}{\Lambda} H_\Lambda + \frac{d}{d\Lambda} \left\{ Z_\Lambda \tilde{F}_\Lambda^{(i)} H_\Lambda \right\} - \tilde{F}_\Lambda^{(i)} \frac{d}{d\Lambda} \{ Z_\Lambda H_\Lambda \} + O(\frac{1}{\Lambda} J_p^2 H), \quad (47)$$

with $c_1 = c_3 = 1$ and $c_2 = c_4 = 0$. All arguments used in Section II B can then be overtaken in the same way, one just has to replace the logarithmic function $\ln \frac{\Lambda + \Gamma}{\Lambda}$ by the functions $\tilde{F}^{(i)}$ in the corresponding terms. In particular this

means that the terms with the integrals $F^{(2,4)}$ do not contribute to the first correction beyond the reference solution since they appear in third order and fall off sufficiently fast $\sim \frac{T^2}{\Lambda^3}$ for $\Lambda \gg T$. Thus, for the conductance, we obtain by analogy with (37) for $\Lambda > T$

$$G(E = i\Lambda, T) = 4x_L x_R \frac{3\pi^2}{4} J_p^2(\Lambda) \left\{ 1 - 4J_p(\Lambda) \tilde{F}_\Lambda^{(1)}(T) + O(J_p^2) \right\}. \quad (48)$$

Following Section II B, in the next regime $\Lambda < T$, we expand in the reference solution $J_p(T)$ evaluated at T . Replacing all vertices on the r.h.s. of (23) by the reference solution at $\Lambda = T$, we see that only the first term involving $F^{(1)}$ can correct the result in $O(J_p^3)$. Furthermore, for the integration $\int_T^0 d\Lambda$, we use the identity $iF_\Lambda^{(1)} = Z_{\Lambda=T} d_\Lambda \hat{F}_\Lambda^{(1)} + O(\frac{1}{T} J_p^2)$, with $\hat{F}_\Lambda^{(1)} = \frac{d}{d\rho} \{\rho \Psi(\rho)\} - 1 - \ln \frac{Z_{\Lambda=T}}{2\pi}$, which can be proven by using $|Z_\Lambda - Z_{\Lambda=T}| \sim O(J_p^2)$, $\frac{d\rho}{d\Lambda} = \frac{1}{2\pi T} (1 + O(J_p^2))$ and $F_\Lambda^{(1)} \sim O(\frac{1}{T})$ for $\Lambda < T$. This form is constructed in such a way that $\hat{F}_\Lambda^{(1)}$ is well-behaved for $\Lambda \rightarrow 0$ and, at the lower boundary $\Lambda = T$ of the integration, is exactly identical to $\tilde{F}_{\Lambda=T}^{(1)}$, such that the third order term of (48) is canceled. The remaining first term of (48) evaluated at $\Lambda = T$ together with the contribution from the upper bound $\Lambda = 0$ of the integration gives for the conductance at $E = 0$ the result $G = 4x_L x_R \frac{3\pi^2}{4} J_p^2(T) \left\{ 1 - 4J_p(T) \hat{F}_{\Lambda=0}^{(1)} + O(J_p^2) \right\}$. Using $\Gamma_{\Lambda=0} \sim O(T J_p^2)$ (which can be derived from an analogous weak coupling expansion) we get $\rho_{\Lambda=0} = \frac{Z_{\Lambda=0} \Gamma_{\Lambda=0}}{2\pi T} \sim O(J_p^2)$, i.e. $\Lambda = 0$ is equivalent to $\rho = 0$ up to higher-order terms. Using $\frac{d}{d\rho} \{\rho \Psi(\rho)\}_{\rho=0} = -\gamma$, where $\gamma = 0.577\dots$ is Euler's constant, and $Z_{\Lambda=T} = 1 + O(J_p)$, we get $\hat{F}_{\Lambda=0}^{(1)} = -\gamma - 1 + \ln(2\pi) + O(J_p)$, leading to the final weak coupling result

$$G(T) = 4x_L x_R \frac{3\pi^2}{4} J_p^2(T) \left\{ 1 - J_p(T) 4(\ln(2\pi) - 1 - \gamma) + O(J_p^2) \right\}. \quad (49)$$

Strong coupling.—In the strong coupling regime $T \ll T_K$ we aim at deriving the Fermi liquid coefficient c_T in the low temperature expansion of the conductance $G = 4x_L x_R \{1 - c_T(T/\bar{\Gamma})^2 + O((T/\bar{\Gamma})^3)\}$ or of its first derivative $\frac{\partial G}{\partial T} = -8x_L x_R \frac{1}{\bar{\Gamma}^2} c_T T + O(T^2/\bar{\Gamma}^3)$. To determine the latter quantity we consider the T -derivative of the RG equations (21-23) and integrate them formally over E (alternatively, one can also derive a diagrammatic technique for T -derivatives similar to the one described in Section I for E - and voltage derivatives⁵). Employing the representation $F^{(i)}(E) = \frac{Z(E)}{T} H^{(i)}(\frac{\chi(E)}{T})$ we find the useful relations

$$\frac{\partial F^{(1/3)}}{\partial T} = -\frac{Z}{T} \frac{\partial}{\partial E} \left(\frac{\chi F^{(1/3)}}{Z} - 1 \right) + O(J^2), \quad \frac{\partial F^{(2/4)}}{\partial T} = -\frac{Z}{T} \frac{\partial}{\partial E} \left(\frac{\chi F^{(2/4)}}{Z} \right) + O(J^2), \quad (50)$$

along with the asymptotic behavior $\chi F^{(1,3)} - Z, \chi F^{(2,4)} \sim O(\frac{T^2}{E^2})$ for $E \gg T$. More precisely, for $E \gg T$, the corrections to (50) are of $O(J^2/E^2)$, such that they can not lead to a reduction of the order in J when integrating the RG equations in the regime from $E = \infty$ to $E = T$. This follows from analyzing the result of the integration procedure, which shows that, for $E \gg T$, we get $\frac{\partial(J_I J_I Z)}{\partial T} \sim \frac{T}{E^2} J^2$, $\frac{\partial K}{\partial T} \sim \frac{T}{E^2} J^3$, and $\frac{\partial \Gamma}{\partial T} \sim \frac{T}{E} J^2$. Furthermore, we made use of $\frac{\partial \chi}{\partial E} = 1 + O(J^2)$. As a consequence, one finds for the T -derivative of (22)

$$\frac{\partial}{\partial E} \frac{\partial(J_I K)}{\partial T} = \frac{6J_I K J Z}{T} \frac{\partial}{\partial E} \left(\frac{\chi F^{(3)}}{Z} - 1 \right) + O(J^5), \quad (51)$$

where the corrections are again of $O(J^5/E^2)$ for $E \gg T$. Integrating (51) over E we obtain to the leading order in J

$$\frac{\partial(J_I K)}{\partial T} = \frac{6J_I K J}{T} (\chi F^{(3)} - Z) + O(J^5). \quad (52)$$

Using this result along with (50) to calculate the T -derivative of (23) we obtain

$$\begin{aligned} \frac{\partial}{\partial E} \frac{\partial G}{\partial T} &= -\frac{6\pi^2 x_L x_R}{T} \left[6J_I K J (\chi F^{(3)} - Z) F^{(1)} - J_I K Z \frac{\partial}{\partial E} \left(\frac{\chi F^{(1)}}{Z} - 1 - 6J \frac{\chi F^{(2)}}{Z} \right) \right] + O(J^5) \\ &= \frac{6\pi^2 x_L x_R}{T} \left\{ \frac{\partial}{\partial E} \left[J_I K (\chi F^{(1)} - Z) - 6J_I K J \chi F^{(2)} \right] + 6J_I K J Z [F^{(1)} - F^{(3)}] \right\} + O(J^5), \end{aligned} \quad (53)$$

where we used $\frac{\partial(J_I K)}{\partial E} = -6J_I K J F^{(3)} + O(J^5)$ and $\frac{\partial Z}{\partial E} \sim O(J^2)$ in the last step. Using (24) and (26) as well as $\frac{d\rho}{dE} = \frac{1}{2\pi i T} + O(J^2)$, we can integrate this result over E and obtain the equation

$$\frac{\partial G}{\partial T} = \frac{6\pi^2 x_L x_R}{T} J_I K \left[(\chi F^{(1)} - Z) - 6J (\chi F^{(2)} - \bar{F}^{(2)}) \right] + O(J^5), \quad (54)$$

where $\bar{F}^{(2)} = Z^2 \left[\frac{d}{d\rho}(\rho\psi(\rho)) - \psi(\rho + \frac{1}{2}) - 1 \right]$ with the asymptotic behavior $\bar{F}^{(2)} \sim O(\frac{T^2}{E^2})$ for $E \gg T$.

Setting $E = 0$ in (54) and using the low temperature expansions $\chi^{F(1)} \rightarrow Z(1 - \frac{1}{6\rho^2})$, $\chi^{F(2)} \rightarrow Z^2 \frac{1}{8\rho^2}$ and $\bar{F}^{(2)} \rightarrow Z^2 \frac{1}{24\rho^2}$ for $\rho = \frac{Z(0)\Gamma(0)}{2\pi T} \gg 1$, we get

$$\frac{\partial G}{\partial T} = -\frac{4\pi^4 x_L x_R \bar{J}_I \bar{K}}{(\bar{Z}\bar{\Gamma})^2} (1 + 3\bar{J}) T + O(T^2, \bar{J}^5) \quad , \quad (55)$$

where $\bar{Z} = Z(0)$, $\bar{\Gamma} = \Gamma(0)$, $\bar{J} = Z(0)J(0)$, $\bar{J}_I = Z(0)J_I(0)$ and $\bar{K} = K(0)$. This expression leads us to the final result for the Fermi liquid coefficient c_T

$$c_T = \frac{\pi^4}{2} \frac{\bar{J}_I \bar{K}}{\bar{Z}^2} (1 + 3\bar{J}) + O(\bar{J}^5) = \frac{\pi^4 \bar{J}^3 (1 + 3\bar{J})}{(1 - \bar{J})^3} + O(\bar{J}^5), \quad (56)$$

where we used $\bar{J}_I = \bar{J}(1 - \bar{J})$, $\bar{K} = 2\bar{J}^2$ and $\bar{Z} = (1 - \bar{J})^2$ [cf. (31)] in the last step. Comparing c_T with the Fermi liquid coefficient c_V [cf. (45)] we get the familiar universal ratio^{9,10} $\frac{c_V}{c_T} = \frac{3}{2\pi^2}$ of the Fermi liquid coefficients in the Kondo model.

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